Kripke-Joyal Semantics

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1 Introduction

1.1 External vs Internal

Throughout this class, we have attempted to find different models of logical theories and analyze their properties as categories. We've also shown dualities between syntax and semantics. However, our underlying languages are always fully syntactic ($\vdash$).

Here, we hope to be able to understand the underlying logic of a theory completely within a category, only speaking w.r.t its objects and arrows.

1.2 Motivation

Textbook first-order semantics treats a model M as a set-based structure and evaluates every sentence externally: a formula is either true or false in M. However, many naturally occurring categories already carry logical structure inside them. Kripke–Joyal (KJ) semantics is the recipe that translates ordinary first-order syntax into this internal logic.

Concretely it replaces statements of the form "$M \vDash \varphi$" with "all generalized elements $c : Z \to [A]^M$ forces φ", which is a fully internal description.

Some examples of this notion is how open sets of a topological space form a Heyting algebra with sheaves over the space witnessing their internal logic. Similarly, in algebraic geometric, working in the étale topos allows for descriptions of "generic points" in scheme theory (bypassing localization arguments).

In other words, KJ semantics internalises the logic to make complicated categorical worlds behave like their familiar classical analogues without losing the information carried by their intrinsic structure.

1.3 Subobjects and membership

We have often discussed the concept of generalized elements in a category $\mathbb{C}$, such that such an "element" of $C \in \mathbb{C}$ is given by some $z : Z \to C$.

Now, as we constructed the concept of a subobject $S \rightarrowtail C$ in order to generalize the $\mathbf{Set}$-based concept of subsets, a natural next step is to define a notion of generalized elements being members of a subobject. Namely:

> **Definition 1.1 (Member of a subobject)**: If we consider a subobject $S \rightarrowtail C$, we can define the say that such a generalized element $z : Z \to C$ is a *member* of this subobject – $z \underset{C}{\in} S$ – iff z factors through the subobject.

In other words, we have that the following commutes:

$$\begin{tikzcd}
 & S \arrow[d, tail] \\
Z \arrow[ur, dashed] \arrow[r, "z"] & C
\end{tikzcd}$$

1.4 Formulas in Context

For $[x : A \mid \varphi]$, for $M \in \mathrm{Mod}(\mathbb{T}, \mathbb{C})$, we interpret it as $[x : A \mid \varphi]^M \rightarrowtail [A]^M$ – in $\mathbf{Sub}([A]^M)$.

We then observe that:

> **Theorem 1.2**: For first order $\mathbb{T}$ and $M \in \mathrm{Mod}(\mathbb{T}, \mathbb{C})$, $M \vDash [x : A \mid \varphi] \iff M \vDash [x : A \mid \top \vdash \varphi]$

Proof:

We know that for such M, M satisfies $[x : A|\varphi]$ iff its interpretation is the maximal subobject. This means $[x : A \mid \varphi]^M \rightarrowtail [A]^M$ is the identity (so $[x : A \mid \varphi] = [A]^M$).

$$\begin{aligned} M \vDash [x : A \mid \varphi] &\iff ([x : A|\varphi]^M \rightarrowtail [A]^M) = \mathrm{id}_{[A]^M} \\ &\iff M \vDash [x : A \mid \top \vdash \varphi] \end{aligned}$$

□

(For concision, write : $([x : A|\varphi]^M \rightarrowtail [A]^M) = \mathrm{id}_{[A]^M} \iff [x : A|\varphi]^M = \mathrm{id}$)

1.5 Factoring

> **Theorem 1.3**: All generalized elements $z : Z \to [A]^M$ factor through $[x : A \mid \varphi]^M \in \mathbf{Sub}([A]^M)$ iff $[x : A|\varphi]^M = \mathrm{id}$.

Proof:

- $(\Longrightarrow)$

If z factors through $[x : A \mid \varphi]^M$, then there exists some $u : [A]^M \to [x : A|\varphi]^M$ such that the following commutes:

$$\begin{tikzcd} & {[x : A \mid \phi]^M} \arrow[d, tail, "v"] \\ {[A]^M} \arrow[ur, dashed, "u"] \arrow[r, tail, "\mathrm{id}"] & {[A]^M} \end{tikzcd}$$

In other words we have that $[A]^M \cong [x : A|\varphi]^M$. But as $[x : A|\varphi]^M \in \mathbf{Sub}([A]^M)$, so $[A]^M = [x : A|\varphi]^M$, so $v = \mathrm{id}$.

- $(\Longleftarrow)$

Suppose $[x : A|\varphi]^M = \mathrm{id}$, so then we have that the following commutes for any generalized element z:

$$\begin{tikzcd} & {[x : A \mid \phi]^M} \arrow[d, equal] \\ Z \arrow[ur, dashed, "z"] \arrow[r, "z"] & {[A]^M} \end{tikzcd}$$

So z trivially factors through $[x : A|\varphi]^M$.

□

1.6 The Forcing Relation

Now, with definintion 2.1 in mind, we now define the forcing relation as an “internal” analogue for entailment. Namely:

> **Definition 1.4 (Forcing)**:
>
> For a Heyting category $\mathbb{C}$, we define the forcing relation "$\Vdash$" such that $Z \Vdash \varphi(z) \iff z \underset{[A]^M}{\in} [x : A \mid \varphi]^M$.

And now combining this definition with earlier results, we get the property that:

> **Theorem 1.5**: For $M \in \mathrm{Mod}(\mathbb{T}, \mathbb{C})$:
>
> $$M \vDash [x : A \mid \varphi] \iff \forall z : Z \to [A]^M, Z \Vdash \varphi(z)$$

Proof: Follows from theorem 2.3 and definition 2.1 □

1.7 The Forcing Relation (general!)

Recall that this notion of forcing is only well-defined over formulas with context $x : A$. For the sake of propriety, we seek to generalize this definition to be valid for all formulas in context $[\Gamma \mid \varphi]$.

> **Definition 1.6 (Forcing')**:
>
> For a Heyting category $\mathbb{C}$, we define the forcing relation "$\Vdash$" such that $Z \Vdash \varphi(\boldsymbol{z}) \iff \boldsymbol{z} \underset{\Pi_{i=1}^n [A_i]^M}{\in} [x_1 : A_1, ..., x_n : A_n \mid \varphi]^M$.
>
> Note that we are writing $\varphi(\boldsymbol{z}) = \varphi(z_1, ..., z_n)$ for $z_i = \pi_i \circ z : Z \to \Pi_{i=1}^n [A_i]^M \to [A_i]^M$.

Moreover, the property given in theorem 2.5 still holds, in that

- $[\cdot \mid \varphi]^M \in \mathbf{Sub}(\mathbf{1})$, and thus $M \vDash [\cdot \mid \varphi] \iff \forall z : Z \to \mathbf{1}, Z \Vdash \varphi \iff !_Z \underset{\mathbf{1}}{\in} [\cdot \mid \varphi]^M$.
- $M \vDash [x_1 : A_1, ..., x_n : A_n \mid \varphi] \iff \forall \boldsymbol{z} : Z \to \Pi_{i=1}^n [A_i]^M, Z \Vdash \varphi(\boldsymbol{z})$.

1.8 Presheaf Categories

Consider $\hat{\mathbb{C}} = \mathbf{Set}^{\mathbb{C}^{\mathrm{op}}}$ and restrict to considering generalized elements of the form $z : yZ \to [A]^M$ (works as well with $\Gamma = [x_i : A_i, ...]$).

> **Theorem 1.7**: The previous result holds if we only consider the representable generalized elements $z : yZ \to [A]^M$.
>
> Namely, we have that:
>
> $$M \vDash [x : A \mid \varphi] \iff \forall z : yZ \to [A]^M, yZ \Vdash \varphi(z)$$

Proof: Sufficient to show that:

$$(\forall z : yZ \to [A]^M, yZ \Vdash \varphi(c)) \implies (\forall w : W \to [A]^M), W \Vdash \varphi(C)$$

Fix $w : W \to [A]^M$. By *Co-Yoneda Lemma* (or Awodey Prop 8.10), we know that W is the colimit of some $\{yR_i\}_{i \in I}$, so $w : W \to [A]^M$, so there exists $\alpha_i : yR_i \to W$ and $yR_i \Vdash \varphi(w \circ \alpha_i)$ for $i \in I$. □

This allows us to lift factorings from the $\{yR_i\}$'s to the colimit.

1.9 Co-Yoneda Lemma

> **Theorem 1.8 (Co-Yoneda Lemma)**: For all $P \in \hat{\mathbb{C}}$, there exists index set I and diagram $R : I \to \mathbb{C}$ such that $P \cong \lim_{\to I}(yR)$.

Proof:

Note we only provide a sketch of the proof, as it is only a helper lemma to justify our focus on representables. Full details are in the Mac Lane book. Alternatively, you can use the Awodey book's Prop 8.10 directly

$I = \int P$ i.e. the category of elements of P – objects (C, p) for $C \in \mathbb{C}, p \in P(C)$ and morphisms $(C', p') \to (C, p)$ are morphisms $u : C' \to C$ with $p \circ u = p'$. Moreover, consider the projection function $\pi_P : \int P \to \mathbb{C}$.

For a functor $A : \mathbb{C} \to \mathbb{E}$ ($\mathbb{E}$ cocomplete), define $R(E) : C \mapsto \mathrm{Hom}(A(C), E)$ and $L(P) = \mathrm{Colim}(\{A \circ \pi_P\})$ and by [Mac Lane I.5.2], $L : \hat{\mathbb{C}} \leftrightarrows \mathbb{E} : R$ are adjoint.

Now let $\mathbb{E} = \hat{\mathbb{C}}$ and $A = y$, so $R(E)(C) = \mathrm{Hom}(y(C), E) \cong E(C)$. Thus by adjoint isomorphism, we get that:

$$P \cong \lim_{\to I}(y \circ \pi_P)$$

□

2 Kripke-Joyal Semantics

2.1 Kripke-Joyal Semantics

Theorem 2.1 (Kripke-Joyal Semantics):

Fix a presheaf category $\hat{\mathbb{C}}$ and $M \in \mathrm{Mod}(\mathbb{T}, \mathbb{C})$ in FOL, formulas $[x : A \mid \varphi]$, $[x : A \mid \psi]$, $[x : A, y : B \mid \chi]$ in $\mathcal{L}(\mathbb{T})$, $C \in \mathbb{C}$ and morphisms $c, c_1, c_2 : yC \to [A]^M$.

Then we have that:

1) $yC \Vdash \top(c)$ always
2) $yC \Vdash \bot(c)$ never
3) $yC \Vdash c_1 = c_2$ iff $c_1 = c_2 : yC \to [A]^M$
4) $yC \Vdash (\varphi \wedge \psi)(c)$ iff $yC \Vdash \varphi(c)$ and $yC \Vdash \psi(c)$
5) $yC \Vdash (\varphi \vee \psi)(c)$ iff $yC \Vdash \varphi(c)$ or $yC \Vdash \psi(c)$
6) $yC \Vdash (\varphi \implies \psi)(c)$ iff $\forall d : yD \to yC, yD \Vdash \varphi(c \circ d)$ implies $D \Vdash \psi(c \circ d)$.
7) $yC \Vdash (\neg\varphi)(c)$ iff for no $d : yD \to yC, yD \Vdash \varphi(c \circ d)$
8) $yC \Vdash (\exists y : B, \chi(\cdot, y))(c)$ iff $\exists c' : yC \to [B]^M, yD \Vdash \chi(c, c')$.
9) $yC \Vdash (\forall y : B, \chi(\cdot, y))(c)$ iff $\forall d : yD \to yC, \forall d' : yD \to [B]^M, yD \Vdash \chi(c \circ d, d')$.

Proof:

We proceed with each individual case as follows.

Base case ($\top, \bot$)

- $yC \Vdash \top(c) \iff c : yC \to [A]^M$ factors through $[x : A \mid \top]^M \in \mathbf{Sub}([A]^M)$. But $[x : A \mid \top]^M$ is always the maximal subobject, and thus id. So c must factor through $[x : A \mid \top]^M$ always.
- $yC \Vdash \bot(c) \iff c$ factors through $[x : A \mid \bot]^M$, which is the (canonical morphism of) initial object $\mathbf{0}$. Thus, $yC \Vdash \bot(c)$ iff there exists $u : yC \to \mathbf{0}$ s.t.

$$\begin{tikzcd} yC \arrow[r, dashed, "u"] \arrow[dr, "c"'] & \mathbf{0} \arrow[d, "!_{[A]^M}"] \\ & {[A]^M} \end{tikzcd}$$

commutes.

Note there exists $!_{yC} : \mathbf{0} \to yC$, so if such u exists, yC is initial. As $\hat{\mathbb{C}}$ is nontrivial, exists C noninitial, so yC noninitial, so $yC \Vdash \bot(c)$ never (for general yC).

Equality

Recall $yC \Vdash \left[x \underset{A}{=} y\right](\langle c_1, c_2 \rangle)$ iff $\langle c_1, c_2 \rangle : yC \to [A]^M \times [A]^M$ factors through $\left[x : A, y : A \mid x \underset{A}{=} y\right]^M \in \mathbf{Sub}([A]^M \times [A]^M)$.

Recall that $\left[x : A, y : A \mid x \underset{A}{=} y\right]^M$ is the equalizer of $[x : A, y : A \mid x]^M = p_1$ and $[x : A, y : A \mid y]^M = p_2$. This is exactly the diagonal $\Delta_{[A]^M}$.

So $yC \Vdash c_1 = c_2$ iff exists u s.t.

$$\begin{tikzcd}
{[x:A, y:A \mid x=y]^M} \arrow[r, "\Delta_{[A]^M}"] & {[A]^M \times [A]^M} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & {[A]^M} \\
yC \arrow[u, dashed, "u"] \arrow[ur, "\langle c_1, c_2 \rangle"'] & &
\end{tikzcd}$$

As $\langle c_1, c_2 \rangle$ factors through the diagonal, we get that $\langle c_1, c_2 \rangle = \langle u, u \rangle$, so $c_1 = c_2$.

Conjunction

$yC \Vdash (\varphi \wedge \psi)(c)$ iff c factors through $[x : A \mid \varphi \wedge \psi]^M = [x : A \mid \varphi]^M \wedge [x : A \mid \psi]^M \in \mathbf{Sub}([A]^M)$.

- ($\Longrightarrow$) Suppose $yC \Vdash (\varphi \wedge \psi)(c)$, so then there exists $u : yC \to \mathrm{dom}([x : A \mid \varphi \wedge \psi]^M)$ such that the following commutes:

$$\begin{tikzcd}
& {[x:A \mid \phi \wedge \psi]^M} \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & {[x:A \mid \psi]^M} \arrow[d, tail] \\
& {[x:A \mid \phi]^M} \arrow[r, tail] & {[A]^M} \\
yC \arrow[uur, dotted, "u"] \arrow[urr, "c"'] & &
\end{tikzcd}$$

Then consider the canonical $a : yC \to [x : A|\varphi]^M$ and $b : yC \to [x : A|\psi]^M$ given by the diagram. Then we have that c factors through a, b. So $yC \Vdash \varphi(c)$ and $yC \Vdash \psi(c)$.

- ($\Longleftarrow$) Suppose $yC \Vdash \varphi(c)$ and $yC \Vdash \psi(c)$. Then by the pullback UMP, there exists u s.t. the following commutes

$$\begin{tikzcd}
yC \arrow[dr, dashed, "u"] \arrow[drr, bend left] \arrow[ddr, bend right] \arrow[ddrr, bend right, "c"] & & \\
& {[x:A \mid \phi \wedge \psi]^M} \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & {[x:A \mid \psi]^M} \arrow[d, tail] \\
& {[x:A \mid \phi]^M} \arrow[r, tail] & {[A]^M}
\end{tikzcd}$$

Then c factors through u, so $yC \Vdash (\varphi \wedge \psi)(c)$.

Disjunction

(*we're a bit loose here on elements of* $\mathbf{Sub}([A]^M)$ *and the domains of these monos…*)

$yC \Vdash (\varphi \vee \psi)(c)$ iff c factors through $[x : A \mid \varphi \vee \psi]^M = [x : A \mid \varphi]^M \vee [x : A \mid \psi]^M \in \mathbf{Sub}([A]^M)$.

Recall that in $\mathbf{Sub}([A]^M)$, meets are given as pullbacks, which is also just the product. Therefore, by duality, we know that joins are given by coproducts. Therefore, we have that $yC \Vdash (\varphi \vee \psi)$ iff c factors through $[x : A \mid \varphi]^M \coprod [x : A \mid \psi]^M \in \mathbf{Sub}([A]^M)$.

Also, by yoneda lemma, we have that $\mathrm{Hom}(yC, U \coprod V) \cong (U \coprod V)(C) \cong U(C) \coprod V(C)$.

- ($\Longrightarrow$) Suppose $yC \Vdash (\varphi \vee \psi)(c)$, so then there exists $u : yC \to \mathrm{dom}([x : A \mid \varphi \vee \psi]^M)$, so note that $u \in \mathrm{Hom}(yC, \mathrm{dom}([x : A \mid \varphi]^M) \coprod \mathrm{dom}([x : A|\psi]^M))$, so we know that u can be represented as some $u' \in \mathrm{dom}([x : A \mid \varphi]^M)(C) \coprod \mathrm{dom}([x : A \mid \psi]^M)(C)$. But this coproduct is just disjoint union, so this just means that $u' \in \mathrm{dom}([x : A \mid \varphi]^M)(C)$ or $u' \in \mathrm{dom}([x : A \mid \psi]^M)(C)$, which is the same as saying that either there exists $a : yC \to [x : A \mid \varphi]^M$ or $b : yC \to [x : A \mid \psi]^M$ such that:

$$
\begin{tikzcd}
 & & & yC \arrow[dlll, bend right, "a"'] \arrow[dll, bend right, "u"'] \arrow[ddll, bend left, "c"] \\
{[x : A|\phi]^M} \arrow[r] \arrow[dr, tail] & {[x : A|\phi \vee \psi]^M} \arrow[d, tail] & {[x : A|\psi]^M} \arrow[l] \arrow[dl, tail] & \\
 & {[A]^M} & &
\end{tikzcd}
$$

commutes, or

$$
\begin{tikzcd}
 & & & yC \arrow[dll, bend right, "u"'] \arrow[dl, "b"] \arrow[ddll, bend left, "c"] \\
{[x : A|\phi]^M} \arrow[r] \arrow[dr, tail] & {[x : A|\phi \vee \psi]^M} \arrow[d, tail] & {[x : A|\psi]^M} \arrow[l] \arrow[dl, tail] & \\
 & {[A]^M} & &
\end{tikzcd}
$$

commutes,

respectively.

Therefore, c factors through either $[x : A|\varphi]^M$ or $[x : A|\psi]^M$, respectively. Thus, $yC \Vdash \varphi(c)$ or $yC \Vdash \psi(c)$.

- ($\Longleftarrow$) Suppose $yC \Vdash \varphi(c)$ or $yC \Vdash \psi(c)$. Then c factors through either $[x : A \mid \varphi]^M$ or $[x : A|\psi]^M$. We now go in the opposite direction of the other case, to get that there then exists $u : yC \to [x : A \mid \varphi]^M \coprod [x : A|\psi]^M$ such that c factors through it. Therefore, c factors through $[x : A \mid \varphi \vee \psi]^M$, so $yC \Vdash (\varphi \vee \psi)(c)$.

Implication

Note that $yC \Vdash (\varphi \Longrightarrow \psi)(c)$ iff c factors through $[x : A \mid \varphi \Longrightarrow \psi]^M = \left([x : A \mid \psi]^M\right)^{[x:A \mid \psi]^M} = \Psi^\Phi \in \mathbf{Sub}([A]^M)$.

($\Longrightarrow$) Suppose c factors through Ψ^Φ via $u \in \mathrm{Hom}(yC, \Psi^\Phi)$, so there exists $\overline{u} : \varepsilon \circ (u \times \mathrm{id}) : yC \times \Phi \to \Psi$. Fix $d : yD \to yC$ and suppose $D \Vdash \varphi(c \circ d)$ (via factor v). Then we get that by the product UMP:

$$
\begin{tikzcd}
 & yD \arrow[dl, "d"'] \arrow[d, dashed, "w"] \arrow[dr, "v"] & \\
yC \arrow[ddr, "c"'] & yC \times \Phi \arrow[l, "p_1"] \arrow[r, "p_2"'] \arrow[d, "\bar{u}"] & \Phi \arrow[ddl, tail] \\
 & \Psi \arrow[d, tail] & \\
 & {[A]^M} &
\end{tikzcd}
$$

So, $c \circ d$ factors through Ψ, so we have that $D \Vdash \psi(c \circ d)$ as desired.

($\Longleftarrow$) Suppose $\forall d : yD \to yC, D \Vdash \varphi(c \circ d) \Longrightarrow D \Vdash \psi(c \circ d)$, so $c \circ d$ factors through Φ. Take pullback of Φ, c to get $(V, d : V \to yC, p : V \to \Phi)$, and using hypothesis (+co-yoneda) get $q : V \to \Psi$.

Now, by product UMP, we get that there exists $u : V \to yC \times \Phi$ and by pullback UMP we get that there exists $u' : yC \times V$ that makes the diagram commute:

$$\begin{tikzcd}
yC \times \Phi \arrow[rr, dashed, "q'"] \arrow[drr, "\pi_2"] \arrow[ddr, "\pi_1"'] & & \Psi \arrow[dd, bend left, "\iota_\Psi"] \\
& V \arrow[ul, <->] \arrow[ur, "q"] \arrow[r, "p"] \arrow[d, "d"] \arrow[dr, phantom, "\lrcorner", very near start] & \Phi \arrow[d, "\iota_\Phi"] \\
& yC \arrow[r, "c"] & {[A]^M}
\end{tikzcd}$$

Moreover, as $q' \in \mathrm{Hom}(yC \times \Phi, \Psi)$, by the definition of the exponential, there exists $\tilde{q'} \in \mathrm{Hom}(yC, \Psi^\Phi)$ such that $\varepsilon \circ \left(\tilde{q'} \times 1_\Phi\right) = q'$. Then we know that $\iota_\Psi \circ \varepsilon \circ \left(\tilde{q'} \times 1_\Phi\right) = \iota_\Psi \circ q' = c \circ \pi_1$. This, by definition, implies that c factors through Ψ^Φ, and thus, $yC \Vdash (\varphi \Longrightarrow \psi)(c)$ as desired.

Negation

Recall that we define $\neg\varphi \equiv (\varphi \Longrightarrow \bot)$. Thus, $C \Vdash \neg\varphi(c)$ iff $C \Vdash (\varphi \Longrightarrow \bot)(c)$.

This happens iff for all $d : yD \to yC, yD \Vdash \varphi(c \circ d)$ implies $D \Vdash \bot(c \circ d)$. But we know this never happens, so that implication is true iff for all $d : yD \to yC$, $\neg(D \Vdash \varphi(c \circ d))$.

Quantifiers Background

Note by [Awodey 3.3.29], for $U \in \mathbf{Sub}([A]^M \times [B]^M)$, $V \in \mathbf{Sub}([A]^M)$, we can define quantifiers pointwise as:

- $\exists_B(U) : C \mapsto \{a \in [A]^M(C) \mid \exists y \in [B]^M(C), (a, y) \in U(C)\}$
- $\forall_B(U) : C \mapsto \{a \in [A]^M(C) \mid \forall h : D \to C, \forall (x, y) \in ([A]^M \times [B]^M)(D), x = [A]^M(h)(a) \Longrightarrow (x, y) \in U(D)\}$

Existential Quantifier

$yC \Vdash \exists y : B, \chi(c, y)$ iff c factors through $[x : A \mid \exists y : B, \chi(\cdot, y)]^M = \exists_B[x : A, y : B \mid \chi]^M$. Taking $U = [x : A, y : B \mid \chi]^M$, we can apply the previous to get that this happens iff c factors through $\exists_B U$

$\Longleftrightarrow$ exists $c' : yC \to [B]^M$ such that for all D, $\langle c, c' \rangle(D) \in U(D) = [x : A, y : B \mid \chi]^M(D)$.

$\Longleftrightarrow$ there exists c' such that $\langle c, c' \rangle$ factors through $[x : A, y : B \mid \chi]^M$

$\Longleftrightarrow$ there exists c' such that $yC \Vdash \chi(c, c')$.

Universal Quantifier

$yC \Vdash \forall y : B, \chi(c, y)$ iff c factors through $[x : A \mid \forall y : B, \chi(\cdot, y)]^M = \forall_B[x : A, y : B \mid \chi]^M$. Taking $U = [x : A, y : B \mid \chi]^M$, we can apply the previous to get that this happens iff c factors through $\forall_B U$

$\Longleftrightarrow$ for all $h : yD \to yC, (a, b) : yD \to [A]^M \times [B]^M$, if $a = c \circ h$ then for all K, $\langle (c \circ h)(K), b(K) \rangle \in U(K)$.

$\Longleftrightarrow$ for all $d : yD \to yC, d' : yD \to [B]^M$, $\langle c \circ h, b \rangle \in U$.

$\Longleftrightarrow \forall d : yD \to yC, \forall d' : yD \to [B]^M, \langle c \circ d, d' \rangle \in [x : A, y : A \mid \chi]^M$.

$\Longleftrightarrow \forall d : yD \to yC, \forall d' : yD \to [B]^M, yD \Vdash \chi(c \circ d, d')$ $\square$

3 Analysis

3.1 IFOL vs FOL

We claim that KJ semantics models internal IFOL, and not full FOL. It is sufficient to show that it is possible in some presheaves for LEM/DN to fail.

For example, simply consider $\mathbb{C} = 2$ w/ objects $0, 1$ and morphism $0 \to 1$. Then $\hat{\mathbb{C}}$ is given by sets $P(0) = \{*\}, P(1) = \emptyset$ with a trivial function $f : \emptyset \to \{*\}$. Let $U \in \mathbf{Sub}(P)$ be maximal, and then consider the formula $\varphi = [U = \emptyset]$.

Then for an element $z : Z \to \mathbf{1}$ (WLOG consider $Z = y(1), z = \mathrm{id}_{y(1)}$), WTS both φ and $\neg\varphi$ are not forced.

- z forces φ iff at $U(1)$ is empty *and* $U(f)(1) = U(0)$ is empty, which is false.
- Similarly, z forces $\neg\varphi$ iff both $U(0), U(1)$ are nonempty, which is false.

As such, we see that in this choice of $\hat{\mathbb{C}}$ and φ, $\hat{\mathbb{C}} \Vdash \varphi \vee \neg\varphi$ is false. Therefore, the LEM does not necessarily hold, and we cannot model full FOL. Thus, KJ semantics models IFOL.

3.2 Application

As we saw in the LEM example, KJ semantics over presheaves encodes how if a property holds at one “stage” (i.e. object), then it must hold in subsequent “stages” (i.e. opposite on outneighbors). With that in mind, one interesting application of KJ semantics is to model intuitionistic linear-time temporal logic.

Consider $\mathbb{C} = (\mathbb{N}, \leq)$ as a poset category, and then we have that $\hat{\mathbb{C}}$ is given by the time indexed family $\{F(n) \mid n \in \mathbb{N}\}$ with restriction maps (backwards in time) $\rho_{m \leq n} : F(n) \to F(m)$ for all $m \leq n$.

Then, to understand generalized elements in $\hat{\mathbb{C}}$, let's work through a specific example. Consider F a presheaf s.t. $F(n) = \{\top, \bot\}$ for all n and restriction maps the identity. Then if we have a generalized element $z : y(n) \to F$, note that $y(n) = \mathrm{Hom}(-, n) = m \mapsto \begin{cases} \{*\} & , m \leq n \\ \emptyset & , \text{else} \end{cases}$. Then z is given as a natural transformation with components $z_m : y(n)(m) \to F(n)$, so if $m \leq n$, $z_m : \{*\} \to \{\top, \bot\}$ and otherwise it is ill-defined. Additionally, the naturality condition forces us to have that $z_m = z_n$ for all $m \leq n$. In other words, $z : y(n) \to F$ is saying that something is true/false for all times up to n. The value at time n fixes all the values in the past to be the same, and leaves the future unconstrained.

More generally, for a generalized element in $\hat{\mathbb{C}}$ given as $z : y(n) \to F$, we have that z represents the state of a time-evolving system up to time n.

Now, we analyze forcing to note that $y(n) \Vdash \varphi(z)$ iff $y(n) \Vdash [\top \implies \varphi](z)$ which occurs iff for all $d : y(m) \to y(n)$, $z \circ d$ factors through the interpretation of φ. This is no different to checking that for all $m \leq n$, $\varphi(z_m)$ is true.

Interestingly, considering $\mathbb{C}' = (\mathbb{N}, \geq)$ and its corresponding $\widehat{\mathbb{C}'}$, we get that the past/future quantifications are all flipped, in that generalized elements are records of the future of this time evolving system starting at time n, and $y(n) \Vdash \varphi(z)$ iff $\varphi(z_m)$ is true for all $m \geq n$. In other words, $y(0) \Vdash \varphi(z)$ simply means $\square\varphi(z)$, i.e. $\varphi(z)$ is “always” true (in temporal logic).

Now, suppose we have $y(n) \Vdash \exists x, \varphi(x)$. This necessarily means (by the previous reasoning and case (8) of KJ semantics) that there exists some $m \geq n$ and corresponding x_m such that $\varphi(x_m)$ holds. In other words, $y(0) \Vdash \exists x, \varphi(x)$ simply means $\diamondsuit\varphi$, i.e. φ is “eventually” true (in temporal logic).

3.3 Extensions

Note that KJ semantics can be extended to many other settings, such as:

- General Heyting categories
- Grothendieck topoi
- Elementary topoi
- G-sets

etc.

For references on these extensions, one can refer to the Mac Lane book.

3.4 References

- S. Mac Lane and I. Moerdijk. Sheaves in Geometry and Logic. A First Introduction to Topos Theory. Springer-Verlag, New York, 1992.
- Awodey, Steve. Category Theory. 2nd ed. Oxford ; Oxford University Press, 2010. Print.